九十四學年第二學期 PHYS2320 電磁學 期中考試題(共兩頁)

[Griffiths Ch. 7-9]　2006/04/11,　10:10am–12:00am, 教師：張存續

記得寫上學號，班別及姓名等。請依題號順序每頁答一題。

◇ Product rule $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

1. (9%, 9%)

(a) Two metal objects #1 and #2 are embedded in weakly conducting material of conductivity σ. Show that the resistance between them is related to the capacitance of the arrangement by $R = \varepsilon_0/\sigma C$.

(b) Suppose you connected a battery between #1 and #2 and charge them up to a potential difference V_0. If you then disconnect the battery, the charge will gradually leak off. Show that $V = V_0 e^{-t/\tau}$, and find the time constant, τ, in terms of ε_0 and σ.

2. (8%, 8%) Find the self-inductance *L* of a solenoid (radius *R*, length *l*, current *I*, and *n* turns per unit length),

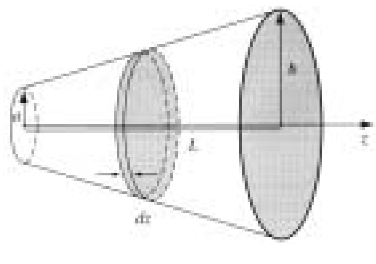
(a) Using the flux relation $\Phi = LI$

(b) Using the energy relation $W = \frac{1}{2}LI^2$.

3. (8%, 8%) Suppose the ends are spherical surfaces, centered at the apex of the cone.

(a) Calculate the resistance in this case. (Let *L* be the distance between the centers of the circular perimeters of the caps.)

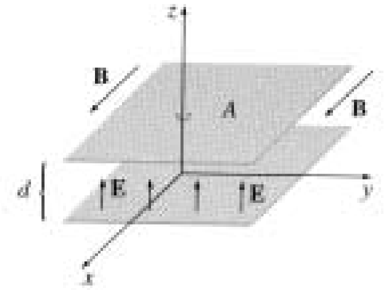
(b) Find the resistance when $a = b$. (Hint: Use $b = a + \Delta$, where Δ approaches zero.)



4. (8%, 8%) A charged parallel-plates capacitor (with uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$) is placed in a uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{x}}$, as shown in the figure.

(a) Find the electromagnetic momentum in the space between the plates.

(b) Suppose we slowly reduce the magnetic field. This will induce a Faraday electric field, which in turn exerts a force on the plates. Show that the total impulse is equal to the momentum originally stored in the field.



5. (8%, 8%) The rate at which work is done on the free charges in a volume V is:

$$\frac{dW}{dt} = \int_V (\mathbf{E}\cdot\mathbf{J}_f)d\tau \text{, where } \mathbf{J}_f = \nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t}.$$

(a) Show that the Poynting vector becomes: $\mathbf{S} = \mathbf{E}\times\mathbf{H}$,

(b) and the rate of change of the energy density in the fields is: $\frac{\partial u_{\text{em}}}{\partial t} = \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}$

6. (8%, 10%)

(a) Write down Maxwell's equations in matter in terms of free charges ρ_f and current $\mathbf{J}_f$,

(b) Write down the equations for conservation of charge, energy, and momentum. Please explain the symbols you use as clear as possible.

1. (a)

$I = \int \mathbf{J} \cdot d\mathbf{a},$ where the integral is taken over a surface enclosing the positive charged conductor.

$$\left.\begin{array}{l} I = \int \mathbf{J} \cdot d\mathbf{a} = \int \sigma \mathbf{E} \cdot d\mathbf{a} = \sigma \int \mathbf{E} \cdot d\mathbf{a} = \dfrac{\sigma}{\varepsilon_0} Q \\ Q = CV \text{ and } V = IR \quad \Rightarrow \quad I = \dfrac{V}{R} = \dfrac{1}{CR} Q \end{array}\right\} R = \frac{\varepsilon_0}{\sigma C}$$

(b)

$$Q = CV = CIR = -CR\frac{dQ}{dt} \quad \Rightarrow \quad Q(t) = Q_0 e^{-t/RC}$$

$$V(t) = \frac{Q(t)}{C} = \frac{Q_0}{C} e^{-t/RC} = V_0 e^{-t/\tau}, \text{ where } \tau = RC = \frac{\varepsilon_0}{\sigma}$$

2. (a) Using the flux relation

$$\left.\begin{array}{l} \Phi_1 = \int \mathbf{B} \cdot d\mathbf{a} = \mu_0 n I \times \pi R^2 = \mu_0 n \pi R^2 I \\ \Phi_{\text{total}} = N\Phi_1 = nl \times \mu_0 n \pi R^2 I = \mu_0 \pi l n^2 R^2 I = LI \end{array}\right. \Rightarrow L = \mu_0 \pi l n^2 R^2 .$$

(b)Using the energy relation

$$W = \int_V \frac{B^2}{2\mu_0} d\tau = \frac{(\mu_0 n I)^2}{2\mu_0} \pi R^2 l = \frac{1}{2}\mu_0 \pi l n^2 R^2 I^2 = \frac{1}{2} L I^2 \quad \Rightarrow \quad L = \mu_0 \pi l n^2 R^2 .$$

3. (a)

$$dR = \frac{\rho}{A} dr, \text{ where } A = \int_0^{2\pi}\int_0^{\theta_0} r^2 \sin\theta d\theta d\phi = 2\pi r^2 (1-\cos\theta_0)$$

$$R = \int dR = \int_{r_a}^{r_b} \frac{\rho}{2\pi r^2 (1-\cos\theta_0)} dr = \frac{\rho}{2\pi(1-\cos\theta_0)} \left.\frac{-1}{r}\right|_{r_a}^{r_b} = \frac{\rho}{2\pi(1-\cos\theta_0)} \left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

$$a = r_a \sin\theta_0 \text{ and } b = r_b \sin\theta_0 \quad \Rightarrow \quad \left(\frac{1}{r_a} - \frac{1}{r_b}\right) = \left(\frac{1}{a} - \frac{1}{b}\right)\sin\theta_0 = \left(\frac{b-a}{ab}\right)\sin\theta_0$$

$$\sin\theta_0 = \frac{b-a}{\sqrt{L^2 + (b-a)^2}} \text{ and } \cos\theta_0 = \frac{L}{\sqrt{L^2+(b-a)^2}}, \quad \frac{1}{(1-\cos\theta_0)} = \frac{\sqrt{L^2+(b-a)^2}}{\sqrt{L^2+(b-a)^2} - L}$$

$$R = \frac{\rho}{2\pi(1-\cos\theta_0)}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) = \frac{\rho}{2\pi} \frac{\sqrt{L^2+(b-a)^2}}{\sqrt{L^2+(b-a)^2} - L} \left(\frac{b-a}{ab}\right) \frac{b-a}{\sqrt{L^2+(b-a)^2}}$$

$$R = \frac{\rho}{2\pi ab} \frac{(b-a)^2}{\sqrt{L^2+(b-a)^2} - L}$$

(b)

Let $b = a + \Delta$

$$R = \lim_{\Delta \to 0} \frac{\rho}{2\pi ab} \frac{\Delta^2}{\sqrt{L^2+\Delta^2} - L} = \frac{\rho}{2\pi a(a+\Delta)} \frac{\Delta^2}{L\left(\dfrac{\Delta^2}{2L^2}\right)} = \frac{\rho L}{\pi a^2}$$

4. (a)

$$\mathbf{g}_{\text{em}} = \varepsilon_0(\mathbf{E}\times\mathbf{B}) = \varepsilon_0 E_0 B_0 \hat{\mathbf{y}} \quad \Rightarrow \quad \mathbf{p}_{\text{em}} = \int_V \mathbf{g}_{\text{em}} d\tau = \varepsilon_0 E_0 B_0 A d \hat{\mathbf{y}}$$

(b)

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\frac{dB}{dt} ld$$

$$-l(E(d)-E(0)) = -\frac{dB}{dt} ld \quad \Rightarrow \quad E(d)-E(0) = d\,\frac{dB}{dt}$$

$$\mathbf{F} = \; -\sigma A(E(d)-E(0))\hat{\mathbf{y}} = \sigma A d\,\frac{dB}{dt}\hat{\mathbf{y}}$$

$$\mathbf{I} = \int_0^\infty \mathbf{F} dt = -\sigma A d(B(t=\infty)-B(t=0))\hat{\mathbf{y}} = \sigma A d B_0 \hat{\mathbf{y}}$$

$$E = \frac{\sigma}{\varepsilon_0} \quad \Rightarrow \mathbf{I} = \sigma A d B_0 \hat{\mathbf{y}} = \varepsilon_0 E_0 B_0 A d \hat{\mathbf{y}} \;\; \text{as before}$$

5. (a) and (b)

$$\frac{dW}{dt} = \int_V (\mathbf{E}\cdot\mathbf{J}_f) d\tau = \int_V (\mathbf{E}\cdot(\nabla\times\mathbf{H}) - \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}) d\tau$$

Product rule: $\mathbf{E}\cdot(\nabla\times\mathbf{H}) = -\nabla\cdot(\mathbf{E}\times\mathbf{H}) + \mathbf{H}\cdot(\nabla\times\mathbf{E})$

Faraday's law: $\nabla\times\mathbf{E} = -\dfrac{\partial\mathbf{B}}{\partial t}$

$$\frac{dW}{dt} = \int_V [-\nabla\cdot(\mathbf{E}\times\mathbf{H}) - (\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} + \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t})] d\tau$$

$$= -\int_S (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{a} - \int_V (\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} + \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}) d\tau$$

➔ $S = \mathbf{E}\times\mathbf{H}$

$$\frac{\partial u_{\text{em}}}{\partial \text{t}} = \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} + \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}$$

6. (a)

$$\nabla\cdot\mathbf{D} = \rho_{\text{f}} \qquad \nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t} = \mathbf{J}_{\text{f}}$$

(b) Conservation of charge

$\dfrac{\partial\rho}{\partial t} = -\nabla\cdot\mathbf{J}$, where ρ is the charge density and $\mathbf{J}$ is the current density.

Conservation of energy

$$\frac{\partial}{\partial t}(u_{\text{mech}} + u_{\text{em}}) = -\nabla\cdot\mathbf{S},$$

where u_{mech} is the mechanical energy density, u_{em} is the electromagnetic energy density, and $\mathbf{J}$ is the Poynting vector.

Conservation of momentum

$$\frac{\partial}{\partial t}(\mathbf{g}_{\text{mech}} + \mathbf{g}_{\text{em}}) = -\nabla\cdot(-\overleftrightarrow{\mathbf{T}}),$$

where $\mathbf{g}_{\text{mech}}$ is the mechanical momentum density, $\mathbf{g}_{\text{em}}$ is the electromagnetic momentum flux density, and $\overleftrightarrow{\mathbf{T}}$ is the Maxwell stress tensor.